LAGRANGIAN EMBEDDINGS IN
THE COMPLEMENT OF SYMPLECTIC HYPERSURFACES

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ABSTRACT

For a given embedded Lagrangian in the complement of a complex hyper-
surface we show the existence of a holomorphic disc in the complement
having boundary on that Lagrangian.

1. Introduction and main results

Let $L \subset \mathbb{C}^n$ be a Lagrangian, i.e., an immersed (real) n-manifold $L \subset \mathbb{C}^n$ such
that the restriction $\omega|_L = 0$ of the symplectic 2-form $\omega = d\theta = \sum_{j=1}^n dx^j \wedge dy^j$
vanishes, where $\theta = \sum_{j=1}^n x^j dy^j$ is the Liouville form and $z^j = x^j + iy^j$ are the
(complex) coordinates on $\mathbb{C}^n$. Hence $\theta|_L$ is a closed 1-form and thus defines an
element in $\mathbf{H}^1(L, \mathbb{R})$. In [2] Gromov constructed nonconstant holomorphic discs
with boundary on any *embedded*, closed Lagrangian in $\mathbb{C}^n$. He concluded that
such a Lagrangian is never exact, i.e., $[\theta|_L] \in \mathbf{H}^1(L; \mathbb{R})$ defines a nonvanishing
class in the de Rham cohomology: the symplectic volume of the disc is positive by
the compatibility of the complex and symplectic structure and so is the integral
of θ over the boundary of the disc by Stokes' theorem. In this paper we specialize
the disc found by Gromov in the presence of a complex hypersurface. Let us first
fix some notation (see e.g. [1]).

Definition 1.1: A complete Riemannian manifold is called **geometrically
bounded** (g.b.) if its sectional curvature is bounded above and the injectiv-
ity radius is bounded below. A submanifold is g.b. if its second fundamental

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form is bounded above and if there are constants ϵ, $C > 0$ such that if two points of the submanifold can be joined by a geodesic of length $l \leq \epsilon$ in the ambient space they can be joined by a geodesic of length Cl in the submanifold.

Note that if the manifold (submanifold) in question is compact, it is, of course, geometrically bounded. Now we are ready to state the main statements of this note.

THEOREM 1.2: (i) *Suppose* $L \cong \mathbb{R} \times \mathbf{S}^1$ *is a g.b. Lagrangian embedding into* $\mathbb{C}^2$ *such that there exists a complex line* H *in its complement homologically linked by* L, *i.e.,* $0 \neq lk(H, .): \mathbf{H}_1(L) \to \mathbb{Z}$. *Then each complex linear function which is non-constant on* H *is unbounded on* L.

(ii) *There is no Lagrangian embedding of the Klein bottle into the complement of a (real) plane in* $\mathbb{C}^2$, *homologically linking it.*

Remark 1.3: (0) The Klein bottle is the only closed surface for which it is unknown whether it admits a Lagrangian embedding into $\mathbb{C}^2$ or not. For this, some progress has been reported by Hofer and Luttinger.

(1) We may construct embedded totally real Klein bottles in the complement of a plane nontrivially linking it. Also note that there is an immersed Lagrangian Klein bottle in the complement of a plane linking it nontrivially (see Exercise 1.4.2. of [1]).

(2) Lalonde proves in [3] that if there exists an embedded Lagrangian Klein bottle in $\mathbb{C}^2$ then the double points of the immersed one in (1) can be suppressed by a symplectic Whitney disc. It follows from Theorem 1.2 that this disc may never lie in the complement of the plane.

All this will follow from the main result of the paper. We will formulate it in a more general setting refining the arguments to specify the holomorphic disc to lie in the complement of a complex hypersurface with boundary on a given Lagrangian in that complement.

THEOREM 1.4: *Let* M^{2n} *be a symplectic manifold and* $\Sigma \subset M$ *be a symplectic hypersurface such that there exists a compatible almost complex structure such that* Σ *is a J-complex hypersurface,* M *and* Σ *are geometrically bounded and there is no non-constant holomorphic sphere in* M *with non-positive intersection index with the hypersurface* Σ *inside* M. *Suppose* $U \subset M$ *is a neighborhood of* Σ *and* $L \subset (M \smallsetminus U) \times \mathbb{C}$ *is a g.b. Lagrangian such that its projection onto the second component is bounded. Then there exists a loop* $\gamma \subset L$ *which bounds a non-constant holomorphic disc in* $(M \smallsetminus \Sigma) \times \mathbb{C}$.

Proof of Theorem 1.2.:   (i) Assume there is a complex linear function f which is bounded on L and non-constant on H. Then the hypotheses of Theorem 1.4 are satisfied (with $M = \ker f$ the splitting $\mathbb{C}^2 \cong \ker f \oplus H$ and $\Sigma = \{0\}$) and in the complement of H there is a disc with boundary on L and non-vanishing symplectic area. However, this contradicts $\mathrm{lk}(H,.) \neq 0$ since the boundary of the disc represents an element of its kernel which is not torsion.

(ii) Let L be a Lagrangian embedding of a closed surface into the complement of a (real) plane. Since for a plane to be symplectic is an open condition we may assume that the plane is symplectic (by replacing it with a nearby plane) and fix a translation invariant, and thus integrable, complex structure on $\mathbb{C}^2$ such that the plane is a complex line $H \subset \mathbb{C}^2$. In the complement of H we find a disc with boundary on L and non-vanishing symplectic area. Thus we obtain an element in the kernel of $\mathrm{lk}(H,.)\colon \mathbf{H}_1(L) \longrightarrow \mathbb{Z}$ which is not torsion. Hence $b_1(L) \geq 2$ and the statement follows.   ∎

There are also consequences in higher dimensions which are interesting from another point of view: Here we have much more space to construct Lagrangians of all kinds, e.g., via a "cabeling" procedure (see Proposition 1.2.3 in [1]). But the following corollary shows that not everything is possible:

COROLLARY 1.5: *Let $L \subset \mathbb{C}^n$ be a g.b. Lagrangian in the complement of a complex hyperplane $H \subset \mathbb{C}^n$ such that L and H are homologically linked. If there exists a complex linear function which is not constant on H and bounded on L then $b_1(L) \geq 2$.*

Proof: The hypotheses of Theorem 1.4 are satisfied: Let f denote the complex linear function then, as above, $M = \ker f$, $\mathbb{C}^n \cong \ker f \oplus H'$, where $H' \subset H$ is a complex line in H transverse to the kernel $\ker(f|_H)$ and $\Sigma = \ker(f|_H)$. This, together with the assumptions on the linking and the existence of the complex linear function, implies the statement as in the proof of Theorem 1.2. If L is closed we may take any real 4-plane by the perturbation argument of that proof. ∎

COROLLARY 1.6: *If L is closed and homologically linked with some real $2(n-1)$-plane, then $b_1(L) \geq 2$. In particular, for $n \geq 3$, any Lagrangian embedding of $\mathbf{S}^{n-1} \times \mathbf{S}^1$ in $\mathbb{C}^n$ (which, e.g., can be constructed from Whitney's Lagrangian immersion of the sphere $\mathbf{S}^{n-1}$ into $\mathbb{C}^{n-1}$ via the cabling mentioned above) is homologically unlinked with any real $2(n-1)$-plane in its complement.*   ∎

2. Proof of Theorem 1.4

We prove the observation refining the arguments used to prove the corresponding result of Gromov [2] (for more details see, e.g., [1]).

Denote by $\pi\colon M \times \mathbb{C} \longrightarrow M$ the projection onto the first component. Fix an $\alpha > 2$. We define

$$\mathcal{F} := \{ u \in \mathbf{C}^\alpha((\Delta, \partial\Delta, 1), (M \times \mathbb{C}, L, p)) \mid u \cong p \text{ in } \pi_2(M \times \mathbb{C}, L, p) \},$$

$$\mathcal{G} := \{ g \in \mathbf{C}^{\alpha-1}(\Delta \times (M \times \mathbb{C}), T^{\mathbb{C}}(M \times \mathbb{C}) \mid \pi_*(g|_{\Delta \times U \times \mathbb{C}}) = 0 \},$$

$$\mathcal{M} := \{ (u, g) \in \mathcal{F} \times \mathcal{G} \mid \overline{\partial} u = g \},$$

where by $\overline{\partial}$ denotes the Cauchy–Riemann operator $\frac{1}{2}(\partial/\partial x + J\partial/\partial y)$ in complex coordinates $z = x + iy$ on the disc.

The linearization

$$D_{(u,g)}\colon \mathbf{C}^\alpha((\Delta, \partial\Delta, 1), (u^*TM \times \mathbb{C}, u^*TL, 0)) \times \mathcal{C}^{\alpha-1}(\Delta, u^*TM \times \mathbb{C})$$
$$\longrightarrow \mathcal{C}^{\alpha-1}(\Delta, u^*TM \times \mathbb{C})$$

of the equation $\overline{\partial} u = g$ is surjective for the following reason: there are sufficient degrees of freedom coming from the second component to perturb the equation at the boundary of the disc. Hence an element in the cokernel would lie in the kernel of a Cauchy–Riemann operator plus a zeroth order term and vanish on the boundary which implies its vanishing. Thus

PROPOSITION 2.1: $\mathcal{M} \subset \mathcal{F} \times \mathcal{G}$ is a Banach submanifold. The natural projection $P\colon \mathcal{M} \longrightarrow \mathcal{G}$ is a Fredholm map of index 0. It is regular at $(p, 0)$.

Note that $P^{-1}(0) = \{(p, 0)\}$ consists of a single point. Pick $h := (0, B)$ according to the splitting $T(M \times \mathbb{C}) \cong \pi^*TM \times \underline{\mathbb{C}}$, $\underline{\mathbb{C}}$ being the trivial complex line bundle, with $B \in \mathbb{C}$ understood as a constant section of it. Connect h to 0 by a path $\gamma \subset \mathcal{G}$ transversal w.r.t. P. Hence $P^{-1}(\gamma)$ is a one-dimensional manifold with boundary consisting of $(p, 0)$ and $P^{-1}(h)$. Via the Cauchy integral or Poisson formula one proves that there is no solution to $\overline{\partial} u = h$ provide B is sufficiently large (see, e.g., [1]). Therefore $P^{-1}(h) = \emptyset$. Thus $P^{-1}(\gamma)$ has to be noncompact.

Let $t_0 := \inf\{t \mid P^{-1}(\gamma(t)) = \emptyset\}$. Pick a sequence $t_n \uparrow t_0 > 0$ and $(u_n, g_n) \in P^{-1}(\gamma(t_n)) \subset \mathcal{M}$ with $\lim_{n\to\infty} g_n \in g \in \mathcal{G}$ which does not converge in $\mathcal{M}$. From [2] (for a longer discussion see also [1]) we therefore know that there is "bubbling": a subsequence converging in the sense of Gromov to a solution of $\overline{\partial} u = g$ with $u\colon (\Delta, \partial\Delta) \longrightarrow (M \times \mathbb{C}, L)$ representing a different homotopy class than the u's considered so far and a (non-empty!) collection of nonconstant

holomorphic discs with boundary on L and nonconstant holomorphic spheres counted with multiplicity. The convergence is such that the homology class of the u_n is preserved. Hence the sum of intersection numbers of these holomorphic curves and u with $\Sigma \times \mathbb{C}$ is zero. But due to Lemma 2.3 the discs have nonnegative intersection number with the hypersurface, for they are holomorphic and the boundary is disjoint from it. Also the spheres have positive intersection number with $\Sigma \times \mathbb{C}$ by assumption.

Thus it suffices to show that u has non-negative intersection number with $\Sigma \times \mathbb{C}$, for this excludes spheres in the collection of holomorphic curves. Hence there *is* a non-constant holomorphic disc with boundary on L and zero intersection index with $\Sigma \times \mathbb{C}$ which implies that it has to lie in the complement.

Let us prove the statement about u. Assume the contrary. Pick a point $q \in u(\Delta) \cap \Sigma \times \mathbb{C}$. We claim that its intersection index is positive. Since $\pi_*(g|_{\Delta \times U \times \mathbb{C}}) = 0$ we obtain a holomorphic map $\pi \circ u \colon u^{-1}(U \times \mathbb{C}) \subset \Delta \to M$ near the preimage of q. The intersection indices of u with $\Sigma \times \mathbb{C}$ at q and $\pi \circ u$ with Σ at $\pi(q)$ agree. Due to Lemma 2.3 the latter is a well-defined (finite!) positive number unless the whole image of that neighborhood lies inside Σ. This cannot happen because then u would map completely into $\Sigma \times \mathbb{C}$. But at least the boundary lies in $L \subset (M \smallsetminus \Sigma) \times \mathbb{C}$.   ∎

Remark 2.2: Theorem 1.2 has a simpler proof directly applying Gromov's original theorem using the fact that $\mathbb{C}^n \smallsetminus \mathbb{C}^{n-1}$ can be symplectomorphically embedded in $(\mathbb{R} \times \mathbf{S}^1) \times \mathbb{C}^{n-1}$.

This situation belongs to the case when the complement of Σ is Stein. Here one may argue with the nonexistence of holomorphic curves touching some (weakly) pseudoconvex boundary from the inside instead. However, the statement is true in a more general context.

Let us finish with a discussion on the positiveness of the intersection indices of an embedded almost complex hypersurface and a (pseudo)holomorphic curve not lying inside the former.

LEMMA 2.3: *Consider an almost complex manifold M. Let $\Sigma \subset M$ be an embedded almost complex hypersurface and $u \colon \Delta \longrightarrow M$ be a (pseudo)holomorphic curve. Then either the image of u is completely contained inside Σ or $u^{-1}(\Sigma) \subset \Delta$ is a discrete set inside the open disc Δ, each point coming with a finite positive intersection index.*

Proof: The statement follows easily as in [4] (from a result of Hartman and Wintner formulated as Theorem 1.1 therein) or [5] (from the similarity princi-

ple). However, it seems that it has not been applied to our situation: a (maybe singular) holomorphic curve intersecting an almost complex hypersurface of any dimension. For completeness we include the argument here which is due to Brian White.

Near an intersection point we may pick coordinates $\mathcal{U} \subset \mathbb{C}^{n-1} \times \mathbb{C}$ of M (containing 0 say) such that $\mathcal{U} \cap \Sigma \subset \mathbb{C}^{n-1} \times \{0\}$ and the almost complex structure has the form

$$J(z,\zeta) = \begin{pmatrix} j(z,\zeta) & 0 \\ O(\zeta)(z,\zeta) & i \end{pmatrix},$$

where j is a family of almost complex structures on $\mathbb{C}^{n-1}$, i is the ordinary complex multiplication, and $O(\zeta)$ is a smooth $2(n-1) \times 2$-matrix-valued function (with real entries) vanishing to the first order with $|\zeta|$ and satisfying $O(\zeta)j + iO(\zeta) = 0$. Now, if $(U,u)\colon \Delta \longrightarrow \mathbb{C}^{n-1} \times \mathbb{C}$ is a pseudo-holomorphic curve through $(0,0)$ then in the coordinate chart it satisfies the following differential equation,

$$\frac{\partial u}{\partial \bar{z}} = O'(u)DU,$$

again with a matrix-valued function $O'(\zeta)$ vanishing to the first order with $|\zeta|$. Differentiating this equation w.r.t. $\partial/\partial z$ we end up with the differential inequality

$$|\Delta u| \leq K(|u| + |Du|)$$

for the u-part of that pseudoholomorphic curve. As in [4] we conclude from Hartman's and Wintner's result that either the whole image of the curve will be contained in the hypersurface or there is a positive integer m such that $u(\xi) = \xi^m + O(|\xi|^{m+1})$, proving the statement of the lemma. Another argument was given by Jean-Claude Sikorav. He concludes the result using the similarity principle at the first order differential equation for u as in his paper [5]. ∎

References

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